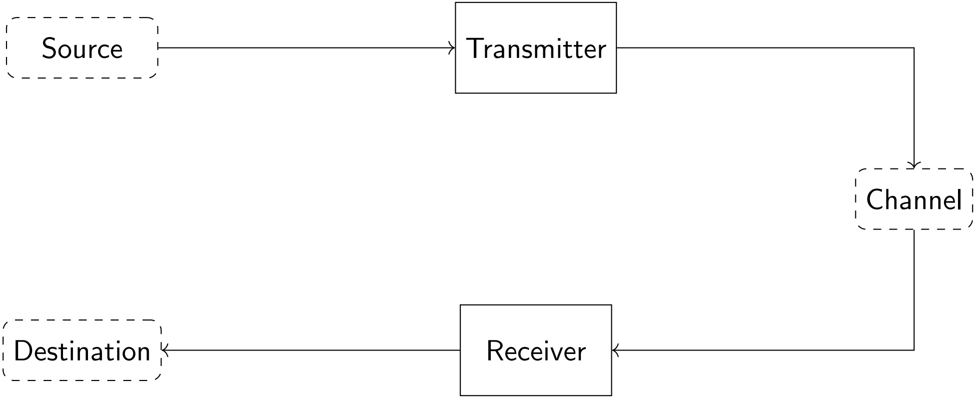
New Paradigms in Source Coding and Channel Coding

Adeel Mahmood

School of Electrical and Computer Engineering
Cornell University

B Exam

Shannon's Source-Channel Separation Theorem



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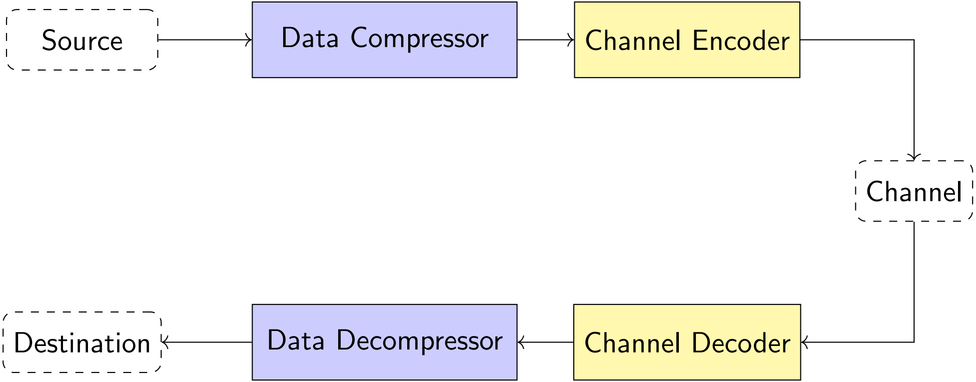
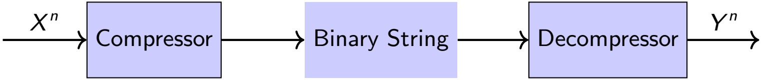


Source Coding



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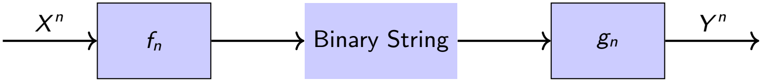

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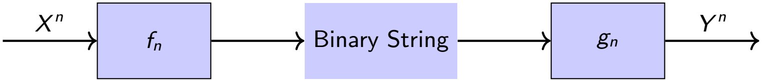

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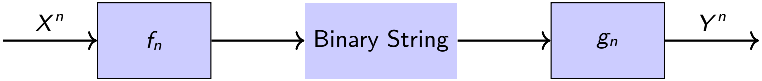
Define the compression rate

$$R(C_n, p) := \frac{\mathbb{E}\left[\mathrm{len}(f_n(X^n))\right]}{n}$$



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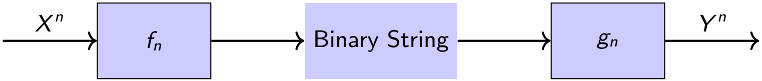

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A compression scheme C_n is ...

- ... <u>non-universal</u> if C_n depends on p and $\lim_{n \to \infty} R(C_n, p) - R(p, d) = 0$

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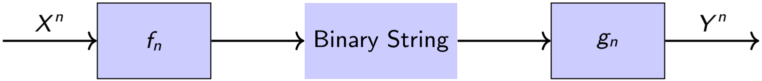

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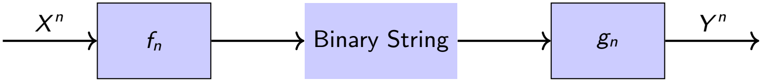

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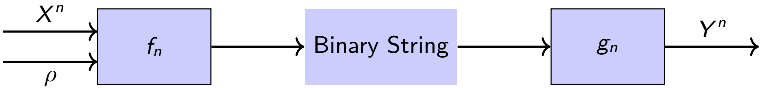

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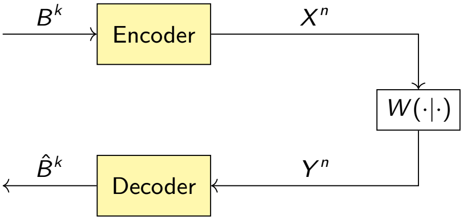
- . . . <u>universal distortion</u> if C_n does not depend on p and ρ and $\lim_{n\to\infty} R(C_n, p) - R(p, d) = 0$

References

- A. Mahmood and A. B. Wagner, "Minimax Rate-Distortion," in IEEE Transactions on Information Theory, 2023
- A. Mahmood and A. B. Wagner, "Lossy Compression With Universal Distortion," in IEEE Transactions on Information Theory, 2023

Channel Coding



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- $W : \mathcal{X} \to \mathcal{P}(\mathcal{Y})$

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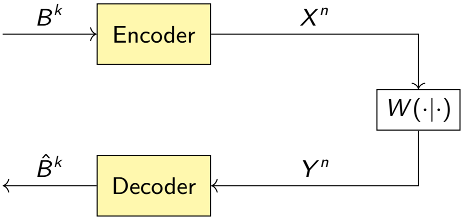

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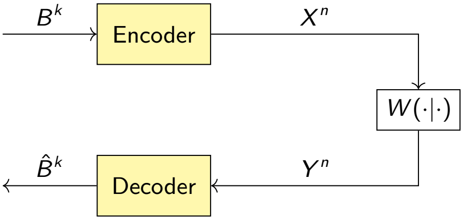

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- Two types of channels:
  - DMC: $\mathcal{X}$ and $\mathcal{Y}$ are finite
  - AWGN: $\mathcal{X} = \mathcal{Y} = \mathbb{R}$ and $W(\cdot|x) = \mathcal{N}(x, 1)$

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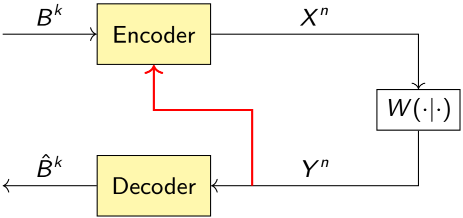

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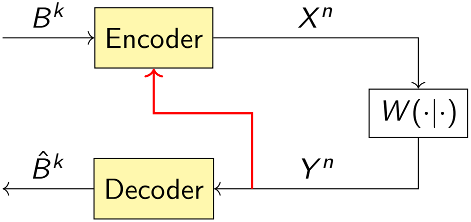
- With feedback, $\text{Enc}(B^k, Y^{m-1}) = X_m$ for all $m = 1, \ldots, n$

The Channel Coding Problem



- Maximize the rate $R = k/n$
- Minimize the probability of error $\Pr(B^k \neq \hat{B}^k)$

Known Results

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Theorem (for DMC):

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$$\text{Let } R = C + \frac{\beta}{\sqrt{n}}$$

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$$\lim_{n\to\infty} P_e\left(n, C + \frac{\beta}{\sqrt{n}}\right) = \Phi\left(\frac{\beta}{\sqrt{V}}\right) \qquad \text{[cf. Strassen '62]}$$

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Theorem (for AWGN):

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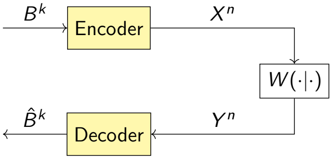
Channel Coding with Cost Constraint



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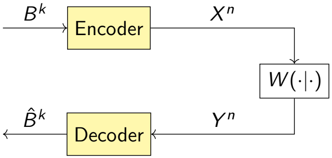

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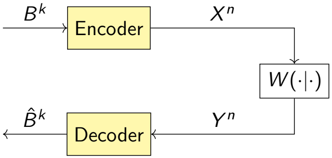

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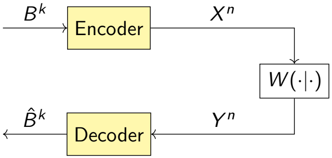

- $c(\cdot) : \mathcal{X} \mapsto [0, \infty)$
- cost threshold: Γ
- **Def** (Almost sure cost constraint): An (n, R, Γ) code is an (n, R) code satisfying

$$\frac{1}{n}\sum_{i=1}^{n} c(X_i) \leq \Gamma \quad \text{a.s.}$$

[Han '98; Hayashi '09; Csiszár and Körner '11; Kostina and Verdú '15]

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$$\lim_{n\to\infty} P_e\left(n, C(\Gamma) + \frac{\beta}{\sqrt{n}}, \Gamma\right) = \Phi\left(\frac{\beta}{\sqrt{V(\Gamma)}}\right) \quad \text{[Hayashi '09]}$$

$$\text{where } C(\Gamma) = \max_{P_X : \mathbb{E}[c(X)] \le \Gamma} I(X;Y) = \mathbb{E}\left[\log \frac{W(Y|X)}{P^* W(Y)}\right]$$

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Limitations of Almost Sure Cost Constraint

$$\frac{1}{n}\sum_{i=1}^{n} c(X_i) \leq \Gamma \text{ a.s.} \quad \text{prohibits codewords generated i.i.d. } P^*,$$

where $P^* = \text{argmax}_{P_X:\mathbb{E}[c(X)]\leq\Gamma} I(X;Y)$.

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> **Theorem 7.7.1** (*Channel coding theorem*) *For a discrete memoryless channel, all rates below capacity C are achievable. Specifically, for every rate $R < C$, there exists a sequence of $(2^{nR}, n)$ codes with maximum probability of error $\lambda^{(n)} \to 0$.*
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> *Conversely, any sequence of $(2^{nR}, n)$ codes with $\lambda^{(n)} \to 0$ must have $R \leq C$.*
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> **Proof:** We prove that rates $R < C$ are achievable and postpone proof of the converse to Section 7.9.
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> *Achievability*: Fix $p(x)$. Generate a $(2^{nR}, n)$ code at random according to the distribution $p(x)$. Specifically, we generate 2^{nR} codewords independently according to the distribution
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> $$p(x^n) = \prod_{i=1}^{n} p(x_i). \qquad (7.60)$$

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Capacity of wireless channels

one picks the codewords randomly and independently, with the components of each codeword i.i.d. $\mathcal{N}(0, P)$, then with very high probability the randomly chosen code will do the job at any rate $R < C$. This is the so-called *i.i.d. Gaussian code*. A sketch of this random coding argument can be found in

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Code construction: We define the common randomness (revealed to the encoder and decoder before the transmission starts) to be a random variable U as follows:

$$P_U \triangleq \underbrace{P_{X^\infty} \times \ldots \times P_{X^\infty}}_{|\mathcal{W}|}, \tag{19}$$

where X^∞ consists of i.i.d. copies drawn from (any) capacity-achieving input distribution. A realization of U defines $|\mathcal{W}|$

In fact, when $\boldsymbol{P}$ is the i.i.d. of P_{M+} or P_{M-}, $I(\epsilon, C \,|\, \boldsymbol{P}, \boldsymbol{W})$ is equal to $\sqrt{V_W^+}F^{-1}(\epsilon)$ or $\sqrt{V_W^-}F^{-1}(\epsilon)$. Thus, (66) holds. Therefore, the achievability part (the direct part) of Theorem 2 holds. Therefore, it is sufficient to prove the converse part (67).

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   2.1 A feedback scheme using i.i.d. and c.c. codewords leads to second-order improvement for DMCs [Mahmood and Wagner '23]
3. Does not allow second-order improvement with feedback in case of unique P^*

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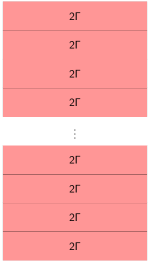

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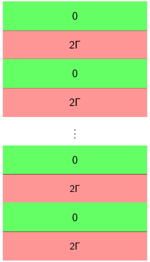

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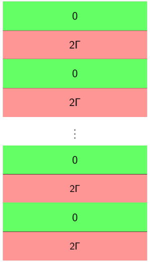

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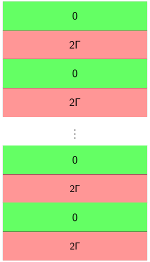

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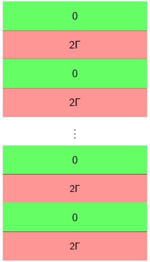

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Def: An (n, R, Γ, Δ) code is an (n, R) code satisfying

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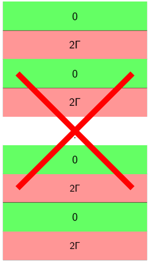

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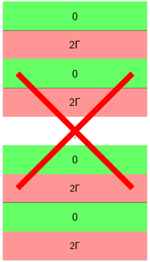

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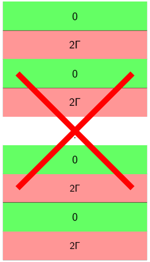

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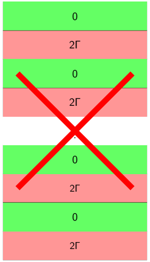

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For any channel with a unique P^*,

$$\lim_{n\to\infty} P_e\left(n, R, \Gamma, \Delta\right) = \begin{cases} 0 & \text{if } R < C(\Gamma) \\ 1 & \text{if } R > C(\Gamma) \end{cases}$$

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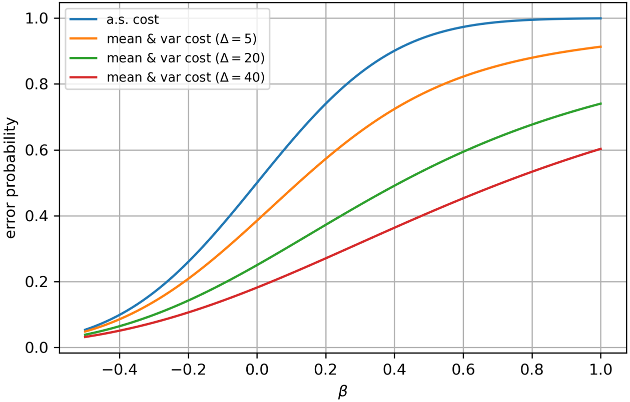
Numerical Example

BSC with crossover prob. $p = 0.3$, $c(x) = x$, $\Gamma = 0.2$, $\epsilon = 0.1$:



Does Feedback Improve Second-Order Performance?

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For any $\Delta > 0$ and a channel (DMC only) with a unique P^*,

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- For unique P^*, feedback does not improve second-order performance [Wagner, Shende, Altuğ '20]

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Proof: Main Ideas

Lemma (cf. Feinstein '54; Shannon '57; Verdú and Han '94)

Let $(X^n, Y^n) \sim P \circ W$.

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Designing a code $\equiv$ **Designing the distribution** P

Proof: Optimal Code Design

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Infimum is over all distributions P such that

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- Choose Q to be capacity-cost achieving output distribution, denoted by Q^*

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Converse Proof

- Let $R = C(\Gamma) + \frac{\beta}{\sqrt{n}}$
- Restrict analysis to $S^* = \left\{x^n : \frac{1}{n}\sum_{i=1}^n c(x_i) \approx \Gamma\right\}$
- Choose Q to be capacity-cost achieving output distribution, denoted by Q^*

$$
\begin{aligned}
P_e &\gtrapprox \inf_P \sup_Q \mathbb{P}\left(\log \frac{W(Y^n|X^n)}{Q(Y^n)} \leq nR\right) \\
&\geq \inf_P \sup_Q \int_{S^*} dP(X^n)\mathbb{P}\left(\log \frac{W(Y^n|X^n)}{Q(Y^n)} \leq nR \middle| X^n = X^n\right) \\
&\geq \inf_P \int_{S^*} dP(x^n)\mathbb{P}\left(\sum_{i=1}^n \log \frac{W(Y_i|x_i)}{Q^*(Y_i)} \leq nR \middle| X^n = x^n\right)
\end{aligned}
$$

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$$P_e \gtrapprox \inf_P \sup_Q \mathbb{P}\left(\log \frac{W(Y^n|X^n)}{Q(Y^n)} \le nR\right)$$

$$\ge \inf_P \sup_Q \int_{S^*} dP(X^n)\mathbb{P}\left(\log \frac{W(Y^n|X^n)}{Q(Y^n)} \le nR \middle| X^n = X^n\right)$$

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$$\text{s.t.} \quad \mathbb{E}[U] \ge \beta$$

$$\mathrm{Var}(U) \le C'(\Gamma)^2\Delta$$

Converse Proof

- Let $R = C(\Gamma) + \frac{\beta}{\sqrt{n}}$
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$$P_e \gtrsim \inf_P \sup_Q \mathbb{P}\left(\log \frac{W(Y^n|X^n)}{Q(Y^n)} \leq nR\right)$$

$$\geq \inf_P \sup_Q \int_{S^*} dP(X^n)\mathbb{P}\left(\log \frac{W(Y^n|X^n)}{Q(Y^n)} \leq nR \middle| X^n = X^n\right)$$

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$$\text{s.t.} \quad \mathbb{E}[U] \geq \beta$$

$$\mathrm{Var}(U) \leq C'(\Gamma)^2\Delta$$

Infimum is over all distributions P such that

$$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n c(X_i)\right] \leq \Gamma \quad \textbf{and} \quad \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^n c(X_i)\right) \leq \frac{\Delta}{n}$$

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\end{aligned}
$$

$\Longrightarrow$

$$
P_e \gtrapprox \inf_U \mathbb{E}\left[\Phi\left(\frac{U}{\sqrt{V(\Gamma)}}\right)\right]
$$

s.t. $\mathbb{E}[U] \geq \beta$

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$$P_e \gtrsim \min_U \ \mathbb{E}\left[\Phi\left(\frac{U}{\sqrt{V(\Gamma)}}\right)\right]$$

s.t. $\mathbb{E}[U] \ge \beta$

$\mathrm{Var}(U) \le C'(\Gamma)^2\Delta$

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Achievability Proof

$$P_e \lessapprox \inf_P \mathbb{P}\left(\log \frac{W(Y^n|X^n)}{PW(Y^n)} \leq nR\right)$$

↑

Infimum is over all distributions P such that

$$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} c(X_i)\right] \leq \Gamma \qquad \textbf{and} \qquad \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} c(X_i)\right) \leq \frac{\Delta}{n}$$

Achievability Proof

1. Let $R = C(\Gamma) + \frac{\beta}{\sqrt{n}}$

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Achievability Proof

1. Let $R = C(\Gamma) + \frac{\beta}{\sqrt{n}}$
2. Choose

$$P = \begin{cases} p_1 \text{Unif}(T^n(t_1)) + p_2 \text{Unif}(T^n(t_2)) + p_3 \text{Unif}(T^n(t_3)) & \text{(for DMC)} \\ p_1 \text{Unif}(S_{R_1}^{n-1}) + p_2 \text{Unif}(S_{R_2}^{n-1}) + p_3 \text{Unif}(S_{R_3}^{n-1}) & \text{(for AWGN)} \end{cases}$$

$$P_e \lessapprox \inf_P \mathbb{P}\left(\log \frac{W(Y^n|X^n)}{PW(Y^n)} \leq nR\right)$$

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Achievability Proof

$$P_e \lesssim \inf_{P} \mathbb{P}\left(\log \frac{W(Y^n|X^n)}{PW(Y^n)} \leq nR\right)$$

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Achievability Proof

1. Let $R = C(\Gamma) + \frac{\beta}{\sqrt{n}}$
2. Choose $P = p_1 \mathrm{Unif}(S_{R_1}^{n-1}) + p_2 \mathrm{Unif}(S_{R_2}^{n-1}) + p_3 \mathrm{Unif}(S_{R_3}^{n-1})$,

$$P_e \lesssim \inf_P \mathbb{P}\left(\log \frac{W(Y^n|X^n)}{PW(Y^n)} \leq nR\right)$$

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   where each $R_i = O(\sqrt{n})$ and $|R_i - R_j| = O(1)$ for $i \neq j$

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Achievability Proof

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$$P_e \lessapprox \mathbb{P}\left(\log \frac{W(Y^n|X^n)}{PW(Y^n)} \le nR\right)$$

$$P_e \gtrapprox \inf_P \sup_Q \mathbb{P}\left(\log \frac{W(Y^n|X^n)}{Q(Y^n)} \le nR\right)$$

↓

$$P_e \gtrapprox \min_U \mathbb{E}\left[\Phi\left(\frac{U}{\sqrt{V(\Gamma)}}\right)\right]$$

s.t. $\mathbb{E}[U] \ge \beta$

$\text{Var}(U) \le C'(\Gamma)^2 \Delta$

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Achievability Proof

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$$\begin{aligned} P_e \gtrapprox \min_U \ & \mathbb{E}\left[\Phi\left(\frac{U}{\sqrt{V(\Gamma)}}\right)\right] \\ \text{s.t.} \quad & \mathbb{E}[U] \geq \beta \\ & \mathrm{Var}(U) \leq C'(\Gamma)^2 \Delta \\ & |\mathrm{supp}(U)| \leq 3 \end{aligned}$$

Achievability Proof

1. Let $R = C(\Gamma) + \frac{\beta}{\sqrt{n}}$
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   where each $R_i = O(\sqrt{n})$ and $|R_i - R_j| = O(1)$ for $i \neq j$

$$P_e \lessapprox \mathbb{P}\left(\log \frac{W(Y^n|X^n)}{PW(Y^n)} \leq nR\right) \neq \mathbb{P}\left(\sum_{i=1}^{n} \log \frac{W(Y_i|X_i)}{PW(Y_i)} \leq nR\right)$$

$$P_e \gtrapprox \inf_P \sup_Q \mathbb{P}\left(\log \frac{W(Y^n|X^n)}{Q(Y^n)} \leq nR\right)$$

$$P_e \gtrapprox \min_U \mathbb{E}\left[\Phi\left(\frac{U}{\sqrt{V(\Gamma)}}\right)\right]$$

s.t. $\mathbb{E}[U] \geq \beta$

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Mixture of Uniform Distributions on $(n-1)$ spheres

- Let $X^n \sim P$, where $$P = p_1 \text{Unif}(S_{R_1}^{n-1}) + p_2 \text{Unif}(S_{R_2}^{n-1}) + p_3 \text{Unif}(S_{R_3}^{n-1}),$$ where each $R_i = O(\sqrt{n})$ and $|R_i - R_j| = O(1)$ for $i \neq j$

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- Let $Y^n \sim PW$, where $PW = p_1 Q_{R_1}^{\text{unif}} + p_2 Q_{R_2}^{\text{unif}} + p_3 Q_{R_3}^{\text{unif}}$

Mixture of Uniform Distributions on $(n-1)$ spheres

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- Let $Y^n \sim PW$, where $PW = p_1 Q_{R_1}^{\text{unif}} + p_2 Q_{R_2}^{\text{unif}} + p_3 Q_{R_3}^{\text{unif}}$
- Would prefer $Q_{R_i}^{\text{i.i.d.}} \longleftarrow$ induced output distribution when the input is $\mathcal{N}\left(\mathbf{0}, \frac{R_i}{\sqrt{n}} I_n\right)$

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Lemma 1 (Mahmood and Wagner '25)

Whether $Y^n \sim Q_{R_i}^{\text{unif}}$ or $Y^n \sim Q_{R_i}^{\text{i.i.d.}}$, for some constant $C > 0$,

$$\mathbb{P}\left(\left|\log \frac{Q_{R_i}^{\text{unif}}(Y^n)}{Q_{R_i}^{\text{i.i.d.}}(Y^n)}\right| \leq C \log n\right) \to 1 \text{ as } n \to \infty.$$

Mixture of Uniform Distributions on $(n-1)$ spheres

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Lemma 2 (Mahmood and Wagner '25)

Whether $Y^n \sim Q_{R_i}^{\mathrm{unif}}$ or $Y^n \sim Q_{R_j}^{\mathrm{unif}}$, for some constant $C > 0$,
$$\mathbb{P}\left(\left| \log \frac{Q_{R_i}^{\mathrm{unif}}(Y^n)}{Q_{R_j}^{\mathrm{unif}}(Y^n)} \right| \leq C \log n \right) \to 1 \text{ as } n \to \infty.$$

Choosing High vs. Low Variance to Minimize $\Pr(X + F(X) \leq C)$

Decision freedom: After observing $X = x$, set $F(x)$ equal to either $Y \sim \mathcal{N}(1, 1)$ or $Z \sim \mathcal{N}(1, 2)$

Choosing High vs. Low Variance to Minimize $\Pr(X + F(X) \leq C)$

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1. $$\min_F \Pr\left(X + F(X) \leq C\right)$$

Choosing High vs. Low Variance to Minimize $\Pr(X + F(X) \leq C)$

Decision freedom: After observing $X = x$, set $F(x)$ equal to either $Y \sim \mathcal{N}(1, 1)$ or $Z \sim \mathcal{N}(1, 2)$

1. $$\min_F \Pr\left(X + F(X) \leq C\right) = \int dx\, p(x) \min_F \Pr(F(x) \leq C - x)$$

Choosing High vs. Low Variance to Minimize $\Pr(X + F(X) \leq C)$

Decision freedom: After observing $X = x$, set $F(x)$ equal to either $Y \sim \mathcal{N}(1,1)$ or $Z \sim \mathcal{N}(1,2)$

1. $$\min_F \Pr\left(X + F(X) \leq C\right) = \int dx\, p(x) \min_F \Pr(F(x) \leq C - x)$$

2. $$\min_F \Pr(F(x) \leq C - x) = \min\left\{F_Y(C - x), F_Z(C - x)\right\}$$

Choosing High vs. Low Variance to Minimize $\Pr(X + F(X) \leq C)$

Decision freedom: After observing $X = x$, set $F(x)$ equal to either $Y \sim \mathcal{N}(1,1)$ or $Z \sim \mathcal{N}(1,2)$

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2. $$\min_F \Pr(F(x) \leq C - x) = \min\left\{F_Y(C - x), F_Z(C - x)\right\}$$
$$= \begin{cases} F_Y(C - x) & \text{if } x \geq x_{\text{th}} \qquad \text{(low variance)} \\ F_Z(C - x) & \text{if } x < x_{\text{th}} \qquad \text{(high variance)} \end{cases}$$

Application to Feedback

- Minimize $\Pr(X + F(X) \leq C)$
- Set $F(x)$ equal to either $Y \sim \mathcal{N}(1, 1)$ or $Z \sim \mathcal{N}(1, 2)$
- $$\min_F \Pr(x + F(x) \leq C) = \begin{cases} F_Y(C - x) & \text{if } x \geq x_{\text{th}} \\ F_Z(C - x) & \text{if } x < x_{\text{th}} \end{cases}$$ (low variance) (high variance)

$$P_e \approx \Pr\left(\log \frac{W(Y^n|X^n)}{PW(Y^n)} \leq nR\right)$$

Application to Feedback

- Minimize $\Pr(X + F(X) \leq C)$
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$$\begin{aligned} P_e &\approx \Pr\left(\log \frac{W(Y^n|X^n)}{PW(Y^n)} \leq nR\right) \\ &\approx \Pr\left(\log \frac{W(Y^{n/2}|X^{n/2})}{PW(Y^{n/2})} + \log \frac{W(Y^{n/2+1:n}|X^{n/2+1:n})}{P_F W(Y^{n/2+1:n}|Y^{n/2})} \leq nR\right) \end{aligned}$$

- Let P_F be the feedback-adjusted distribution for the second half

Application to Feedback

- Minimize $\Pr(X + F(X) \leq C)$
- Set $F(x)$ equal to either $Y \sim \mathcal{N}(1,1)$ or $Z \sim \mathcal{N}(1,2)$
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- Let P_F be the feedback-adjusted distribution for the second half
- If $\log \frac{W(Y^{n/2}|X^{n/2})}{PW(Y^{n/2})} \leq$ threshold, then choose $P_F = p_1 \text{Unif}(S_{R_1}^{n-1}) + p_2 \text{Unif}(S_{R_2}^{n-1}) + p_3 \text{Unif}(S_{R_3}^{n-1})$

Application to Feedback

- Minimize $\Pr(X + F(X) \leq C)$
- Set $F(x)$ equal to either $Y \sim \mathcal{N}(1,1)$ or $Z \sim \mathcal{N}(1,2)$
- $$\min_F \Pr(x + F(x) \leq C) = \begin{cases} F_Y(C-x) & \text{if } x \geq x_{\text{th}} \\ F_Z(C-x) & \text{if } x < x_{\text{th}} \end{cases} \qquad \begin{matrix} \text{(low variance)} \\ \text{(high variance)} \end{matrix}$$

$$P_e \approx \Pr\left(\log \frac{W(Y^n|X^n)}{PW(Y^n)} \leq nR\right)$$
$$\approx \Pr\left(\log \frac{W(Y^{n/2}|X^{n/2})}{PW(Y^{n/2})} + \log \frac{W(Y^{n/2+1:n}|X^{n/2+1:n})}{P_F W(Y^{n/2+1:n}|Y^{n/2})} \leq nR\right)$$

- Let P_F be the feedback-adjusted distribution for the second half
- If $\log \frac{W(Y^{n/2}|X^{n/2})}{PW(Y^{n/2})} \leq$ threshold, then choose $P_F = p_1 \text{Unif}(S_{R_1}^{n-1}) + p_2 \text{Unif}(S_{R_2}^{n-1}) + p_3 \text{Unif}(S_{R_3}^{n-1})$
- If $\log \frac{W(Y^{n/2}|X^{n/2})}{PW(Y^{n/2})} >$ threshold, then choose $P_F = \text{Unif}(S_R^{n-1})$

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